Causality Violation and Non-local Generalization of the Lorentz-Dirac Equation

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It is demonstrated by a concrete example (constant force of finite duration) that the recently proposed, non-local equation of motion for the radiating electron does exhibit the effect of causality violation. This phenomenon, which occurs in the non-local theory in form of self-oscillations, is however less severe than in the Lorentz-Dirac theory, if only physically reasonable forces are admitted.

I. Introduction and Survey of Results

It is a well-known fact in the literature of classical radiation reaction theories that there are some problems in this field, which never have been solved satisfactorily. It is just the most famous one of those theories, namely the Lorentz-Dirac theory [1] based upon the equation *

$$m c^2 \dot{u}^\lambda = K^\lambda + \tfrac{2}{3} Z^2 \{\ddot{u}^\lambda + (\dot{u}\,\dot{u})\,u^\lambda\}, \quad \text{(I, 1a)}$$

which demonstrates clearly the sort of troubles encountered here: Since the usual equation of motion for a non-radiating particle

$$m c^2 \dot{u}^\lambda = K^\lambda \quad \text{(I, 2)}$$

has been supplemented by the higher-derivative term Γ^λ

$$\Gamma^\lambda = \tfrac{2}{3} Z^2 \{\ddot{u}^\lambda + (\dot{u}\,\dot{u})\,u^\lambda\} \quad \text{(I, 3)}$$

the covariant third-order differential Eq. (I, 1a) requires a new initial (resp. final) condition in addition to position and velocity. Moreover, the acceleration is different from zero before a force is switched-on (preacceleration), and the acceleration at a certain time s is determined by all future forces, as it is best seen in the case of one-dimensional motion [1]

$$m c^2 \dot{w}_{(s)} = \int_{s'=s}^{\infty} \exp\left\{-\frac{s'-s}{\Delta s}\right\} K_{(s')} \frac{ds'}{\Delta s}. \quad \text{(I, 1b)}$$

The "derivation" of Eq. (I, 1) from Maxwell's theory of electromagnetism is a problem in itself, but with this problem we are not concerned here.

But also other equations of motion for the classically radiating electron exhibit serious disadvantages: the finite-differences theory of Caldirola [2]

$$m c^2 \left\{ \frac{u^\lambda_{(s)} - u^\lambda_{(s-s_0)}}{s_0} - \left[u^\nu_{(s)} \frac{u_{\nu(s)} - u_{\nu(s-s_0)}}{s_0}\right] u^\lambda_{(s)} \right\} = K^\lambda \quad \text{(I, 3)}$$

admits undetermined self-oscillations (put $u^\lambda_{(s)} = u^\lambda_{(s-s_0)}$ in (I, 3) for $K^\lambda \equiv 0$). Or the equation of Mo and Papas [3]

$$m c^2 \dot{u}^\lambda + \frac{2}{3} \frac{Z^3}{m c^2} (F^{\mu\nu} \dot{u}_\mu u_\nu) u^\lambda = Z F^{\mu\lambda} u_\mu + \frac{2}{3} \frac{Z^3}{m c^2} F^{\mu\lambda} \dot{u}_\mu \quad \text{(I, 4)}$$

reduces to the neutral particle limit (I, 2) in the case of one-dimensional motion, which is excusable only if this equation of motion is considered as an approximative one [4]. And higher-derivative theories, such as that (e. g.) of Eliezer [5], exhibit a high degree of arbitrariness; they are not able to exclude runaway solutions and make the initial-value problem more complicated.

As a way out of all these troubles we have proposed ** the following equation of motion for the radiating electron [6]

$$m c^2 \{\dot{u}^\lambda_{(s-\Delta s)} - (\dot{u}^\nu_{(s-\Delta s)} u_{\nu(s)}) u^\lambda_{(s)}\} = Z F^{\mu\lambda}_{(s-\Delta s)} u_{\mu(s)} \quad \text{(I, 5a)}$$

* The notation is here the same as in the preceding papers [4, 6]: four-velocity $c\,u^\lambda$ with $u^\lambda u_\lambda = +1$, $\dot{u}^\lambda \equiv du^\lambda/ds$ etc.
** Compare the quite similar ideas of Gromes and Petzold [8].

or, in its one-dimensional form

$$m c^2 \dot{w}_{(s)} \operatorname{Cosh}\left[w_{(s+\Delta s)} - w_{(s)}\right] = K_{(s)} . \qquad \text{(I, 5b)}$$

Indeed, this is a very simple equation of motion containing only one arbitrary parameter (Δs). This is fixed by the requirement

$$m c^2 \Delta s = \tfrac{2}{3} Z^2 , \qquad \text{(I, 6)}$$

which guarantees in a local approximation the emergence of the standard radiation formulae, such as the Larmor radiation rate [4]

$$\mathfrak{R} = \frac{1}{c^2} \frac{dE_{\text{rad}}}{dt} = -\frac{1}{c} \frac{2}{3} Z^2 (\dot{u}\, \dot{u}) \qquad \text{(I, 7a)}$$

or the radiated four-momentum

$$\frac{dP_{\text{rad}}^\mu}{ds} = -\frac{1}{c} \frac{2}{3} Z^2 (\dot{u}\, \dot{u})\, u^\mu . \qquad \text{(I, 7b)}$$

The new Eq. (I, 5) avoids the above listed disadvantages of the other theories: especially the old simplicity of initial conditions (position and velocity) is restored [7]. However, there is one exception: this is the phenomenon of causality violation. In Ref. [7] we have demonstrated under what conditions the future forces determine the earlier values of the acceleration, and in the present paper we intend to study this last disadvantage, which is left in the new theory after all other disadvantages have been successfully eliminated.

To this end, we proceed in three steps: First, we investigate the strength of causality violation in the general one-dimensional case (I, 5b) and we shall find that causality violation in the new non-local theory (I, 5b) is much less severe than in the Lorentz-Dirac case (I, 1b), if the external force $|K|$ is less than the interaction force of two Coulomb singularities in the distance of (roughly) two times the classical electron radius

$$r_c = \frac{Z^2}{2 m c^2} \approx 1.4 \cdot 10^{-13}\,\text{cm} .$$

This condition is surely satisfied in physically reasonable situations referring to the classical domain. However, if the force $|K|$ is greater than the critical value just mentioned, then the causality violation effect is more severe than in the Lorentz-Dirac case (Section II).

Next, a numerical solution of the constant-force problem (of finite duration) is presented, which clearly demonstrates the effect of causality violation in the form of "self-oscillations", which are excited whenever the temporal end of the force approaches: the particle can "feel" the end of the force in advance (Section III)!

Finally, the self-oscillations are treated in a linearized way, and they are found to be of the form $e^{as} \sin(b\,s)$. The dependence of the "damping constant" $a_{(K)}$ and the period $b_{(K)}$ upon the force K is discussed numerically. For vanishing force $(K \to 0)$, the damping constant becomes infinite, which means that causality violation is not present in this limit; but for $K \to \infty$ the causality violating effects reach back in time up to the infinitely distant past $(a \to 0)$ (Section IV).

II. General Limits for Causality Violation in the One-dimensional Case

In this section we intend to study the influence of the future forces on the invariant acceleration $\dot{w}_{(s)}$ at present time. In order to do this, we compare the motion of the particle in two force fields, which differ from each other not until a certain proper time s_{fin}; i.e. we apply the equation of motion (I, 5b) to the following force problems (abbreviate $F_{(s)} := K_{(s)}/m c^2$):

$$F_{(s)} = \begin{cases} F_{0(s)}; & s \leqq s_{\text{fin}} , \\ G_{(s)}; & s > s_{\text{fin}} , \end{cases} \qquad \text{(II, 1a)}$$

and

$$F_{(s)} = \begin{cases} F_{0(s)}; & s \leqq s_{\text{fin}} , \\ H_{(s)}; & s > s_{\text{fin}} , \end{cases} \qquad \text{(II, 1b)}$$

where $F_{0(s)}$, $G_{(s)}$ and $H_{(s)}$ are known force functions of proper time s, and $H_{(s)}$ is not identical to $G_{(s)}$. Now we ask what are the maximal deviations of the corresponding accelerations $\dot{w}_{G(s)}$ and $\dot{w}_{H(s)}$ during the common force $F_{0(s)}$ is acting on the particle $(s \leqq s_{\text{fin}})$?

To answer this question, one concludes first from the equation of motion

$$\dot{w}_{G,H(s)} = \frac{F_{0(s)}}{\operatorname{Cosh} \Delta w_{G,H(s)}} , \qquad \text{(II, 2)}$$

which is fulfilled equally for both cases in the proper time period under consideration, that the following inequality holds:

$$\begin{aligned} |\dot{w}_{G(s)} - \dot{w}_{H(s)}| &\leqq m_F \\ &\cdot \left| \frac{1}{\operatorname{Cosh} \Delta w_{G(s)}} - \frac{1}{\operatorname{Cosh} \Delta w_{H(s)}} \right| . \end{aligned} \qquad \text{(II, 3)}$$

Here we have abbreviated the absolute maximum of the common force $F_{0(s)}$ by m_F

$$m_F := \max \left\{ \left| F_{0(s)} \right| \right\}, \qquad (II, 4)$$

assuming that m_F is a finite number, the upper limit of which shall be restricted further in the course of the following considerations.

Moreover, one can use in formula (II, 3) the inequality

$$\left| \frac{1}{\operatorname{Cosh} \Delta w_G} - \frac{1}{\operatorname{Cosh} \Delta w_H} \right| \leqq \tfrac{1}{2} \left| \Delta w_G - \Delta w_H \right|, \qquad (II, 5)$$

thus obtaining

$$\left| \dot{w}_{G(s)} - \dot{w}_{H(s)} \right| \leqq \tfrac{1}{2} m_F \left| \Delta w_{G(s)} - \Delta w_{H(s)} \right|. \qquad (II, 6)$$

Further conclusions concerning the right-hand side of (II, 6) can be drawn by converting the differential-difference equation of motion

$$\dot{w}_{(s)} = F_{0(s)} / \operatorname{Cosh} \Delta w_{(s)} \qquad (II, 7)$$

into an integral equation ($\Phi_{(s)} := \Delta w_{(s)}$)

$$\int_{s'=s}^{s+\Delta s} \dot{w}_{(s')} \, \mathrm{d}s' \equiv \Phi_{(s)} = \int_{s'=s}^{s+\Delta s} \mathrm{d}s' \frac{F_{0(s')}}{\operatorname{Cosh} \Phi_{(s')}}. \qquad (II, 8)$$

This integral equation is satisfied by both $\Phi_{G(s)}$ and $\Phi_{H(s)}$. Hence, we have

$$\left| \Phi_{G(s)} - \Phi_{H(s)} \right| \leqq m_F \int_s^{s+\Delta s} \mathrm{d}s'$$

$$\cdot \left| \frac{1}{\operatorname{Cosh} \Phi_{G(s')}} - \frac{1}{\operatorname{Cosh} \Phi_{H(s')}} \right| \qquad (II, 9)$$

$$\leqq \tfrac{1}{2} m_F \int_s^{s+\Delta s} \mathrm{d}s' \left| \Phi_{G(s')} - \Phi_{H(s')} \right|.$$

Arrived at this point, it is useful to remember the range of s in $\Phi_{(s)} = w_{(s+\Delta s)} - w_{(s)}$: The maximal value of s' in (II, 9) is obviously $s + \Delta s$; therefore one must know $w_{(s+2\Delta s)}$ in order to compute the integral. If we now iterate Eq. (II, 9) n times

$$\left| \Phi_{G(s)} - \Phi_{H(s)} \right| \leqq \left(\tfrac{1}{2} m_F \right)^n$$

$$\cdot \int_{s_1=s}^{s+\Delta s} \mathrm{d}s_1 \int_{s_2=s_1}^{s_1+\Delta s} \mathrm{d}s_2 \dots \int_{s_n=s_{n-1}}^{s_{n-1}+\Delta s} \mathrm{d}s_n \left| \Phi_{G(s_n)} - \Phi_{H(s_n)} \right| \qquad (II, 10)$$

one recognizes at once that the range of s involved in the n-fold integral reaches from s to $s^* = s + (n+1)\Delta s$. But in order that the evaluations being performed up to now are applicable, it was necessary to deal with the common force $F_{0(s)}$. Since this common force ends at $s = s_{\mathrm{fin}}$, where

one has to consider two distinct forces ($G_{(s)}$ and $H_{(s)}$), we have the following requirement on the maximal number N of allowed iterations

$$s^* = s + (N+1)\Delta s \leqq s_{\mathrm{fin}} + \Delta s, \qquad (II, 11a)$$

i. e.

$$\frac{s_{\mathrm{fin}} - s}{\Delta s} - 1 < N \leqq \frac{s_{\mathrm{fin}} - s}{\Delta s}. \qquad (II, 11b)$$

Before we make use of this result, we perform a final evaluation in (II, 10) by putting

$$\left| \Phi_{G(s_n)} - \Phi_{H(s_n)} \right| \leqq \left| \Phi_{G(s_n)} \right| + \left| \Phi_{H(s_n)} \right| \leqq 2 \, m_F \, \Delta s, \qquad (II, 12)$$

which readily follows from the integral Eq. (II, 8), thus obtaining finally (assume $m_{G,H} \leqq m_F$)

$$\left| \Phi_{G(s)} - \Phi_{H(s)} \right| \leqq 4 \left(\tfrac{1}{2} m_F \, \Delta s \right)^{N+1}, \qquad (II, 13)$$

or with (II, 6)

$$\Delta s \left| \dot{w}_{G(s)} - \dot{w}_{H(s)} \right| \leqq 4 \left(\tfrac{1}{2} m_F \, \Delta s \right)^{N+2}. \qquad (II, 14)$$

So we see that if the condition

$$\left(\tfrac{1}{2} m_F \, \Delta s \right) < 1 \qquad (II, 14')$$

on the forces F_0, G, and H is satisfied, then the difference $\left| \dot{w}_G - \dot{w}_H \right|$ is rapidly damped down looking back in time from $s = s_{\mathrm{fin}}$. Using (II, 11b), this damping phenomenon is described by

$$\Delta s \left| \dot{w}_G - \dot{w}_H \right| \leqq 4 \left(\frac{m_F \Delta s}{2} \right)^{(s_{\mathrm{fin}} - s)/\Delta s}. \qquad (II, 15)$$

Condition (II, 15) means for the ordinary force

$$K_{(s)} = m \, c^2 \, F_{(s)} = \frac{2}{3} \frac{Z^2}{\Delta s} F_{(s)}$$

$$\left| K \right|_{\max} < \frac{4}{3} \frac{Z^2}{\Delta s^2}, \qquad (II, 16)$$

i. e. $\left| K \right|_{\max}$ must be smaller than the static contact force of two finite-size electrons, which is, of course, fulfilled for all forces accessible to a classical measurement. Since in experimentally relevant cases $\left| K \right|_{\max} \ll \tfrac{4}{3} Z^2 / \Delta s^2$, we find an extraordinary small causality-violating effect.

It remains to be shown now that for weak fields ($\left| K \right| \ll Z^2 / \Delta s^2$) the extent of causality violation in the present non-local case is much less than in the Lorentz-Dirac case. To this end, we write down the corresponding expressions following from the Lorentz-Dirac equation (I, 1b)

$$\dot{w}_{G(s)} = \int\limits_{s'=s}^{s_{\text{fin}}} \exp\left\{-\frac{s'-s}{\Delta s}\right\} F_{0(s')} \frac{\mathrm{d}s'}{\Delta s} + \int\limits_{s'=s_{\text{fin}}}^{\infty} \exp\left\{-\frac{s'-s}{\Delta s}\right\} G_{(s')} \frac{\mathrm{d}s'}{\Delta s} \tag{II, 17}$$

where G can be replaced by H for the second case. Subtracting these two equations yields ($s < s_{\text{fin}}$)

$$\dot{w}_{G(s)} - \dot{w}_{H(s)} = \exp\left\{-\frac{s_{\text{fin}}-s}{\Delta s}\right\} \cdot \int\limits_{s'=s_{\text{fin}}}^{\infty} \exp\left\{-\frac{s'-s_{\text{fin}}}{\Delta s}\right\} (G_{(s')} - H_{(s')}) \frac{\mathrm{d}s'}{\Delta s} . \tag{II, 18}$$

The integral occurring here is a fixed number so that the causality violating effects fall always down as $\exp\{-(s_{\text{fin}}-s)/\Delta s\}$ if one looks backward in time from $s = s_{\text{fin}}$. This damping factor is independent from the forces applied, whereas formula (II, 15) says that the corresponding damping factor of the non-local theory

$$\left(\exp\left\{\ln\left[\frac{m_F \Delta s}{2}\right] \cdot \frac{s_{\text{fin}}-s}{\Delta s}\right\}\right)$$

depends very strongly from the applied force. For instance, if

$$\frac{1}{2} m_F \Delta s = \frac{|K|_{\max}}{\frac{4}{3} Z^2/\Delta s^2} \stackrel{!}{\approx} 10^{-2}, \tag{II, 19}$$

which means that the maximal force $|K|_{\max}$ equals roughly the interaction force of two Coulomb singularities in the distance of ten times the classical electron radius, then the damping factor of the non-local theory is (roughly) $\exp\{-5(s_{\text{fin}}-s)/\Delta s\}$, which implies a far less severe violation of causality than that occurring in the Lorentz-Dirac theory. We shall elaborate this point in greater detail in the following sections, where the causality-violating behaviour of the electron is studied numerically and in a linearized approach.

III. Solution of the Constant-Force Problem

The one-dimensional equation of motion (II, 7) was solved numerically for the following force $F_{(s)}$:

$$F_{(s)} = \begin{cases} 0; & s < 0, \\ f; & 0 \le s < s_{\text{fin}} = 6 \Delta s, \\ 0; & s_{\text{fin}} \le s. \end{cases} \tag{III, 1}$$

Figures (1 a – d) demonstrate clearly the following two phenomena:

a) Retarded Response to the Switching

The discontinuity of the force at $s = 0$ is reproduced instantaneously by the acceleration $\dot{w}_{(s)}$ in a weakened manner. This weakening is ascribed to radiation reaction, which prohibits the radiating particle to assume the same acceleration as a neutral (i. e.: non-radiating) particle, for which one would have

$$\dot{w}^{\text{np}}_{(s)} = F_{(s)} . \tag{III, 2}$$

At first sight, the instantaneous response of the acceleration to the external force seems to pose a paradox: From formulae (V, 1) and (V, 4a) of the earlier paper [4] one finds for the work of the external force done on the particle per unit laboratory time

$$\frac{\mathrm{d}A_{(s(t))}}{\mathrm{d}t} = \mathbf{K}_{(s(t)-\Delta s)} \cdot \mathbf{v}_{(s(t))} = c\, K_{(s(t)-\Delta s)} \operatorname{Tanh} w_{(s(t))} \tag{III, 3}$$

respectively for the radiated energy

$$\frac{\mathrm{d}E_{\text{rad}(s(t))}}{\mathrm{d}t} = -m\,c^3\,(u\,\hat{u}) = m\,c^3\,\dot{w}_{(s(t)-\Delta s)} \operatorname{Sinh}\left[w_{(s(t))} - w_{(s(t)-\Delta s)}\right], \tag{III, 4}$$

or by use of the equation of motion (II, 7)

$$\frac{\mathrm{d}E_{\text{rad}(s(t))}}{\mathrm{d}t} = c\, K_{(s(t)-\Delta s)} \operatorname{Tanh}\left[w_{(s(t))} - w_{(s(t)-\Delta s)}\right] . \tag{III, 4'}$$

Now, consider a certain lab time t^*, so that $0 < s(t^*) < \Delta s$. Then we know that the particle is accelerated at that time t^*; but on the other hand, it follows from (III, 3) and (III, 4) that

$$\left.\frac{\mathrm{d}A_{(s(t))}}{\mathrm{d}t}\right|_{t=t^*} = \left.\frac{\mathrm{d}E_{\text{rad}(s(t))}}{\mathrm{d}t}\right|_{t=t^*} = 0, \tag{III, 5}$$

because of $K_{(s(t^*)-\Delta s)} = 0$ by virtue of (III, 1). So we see that the particle is accelerated at that time t^* but nevertheless energy is not radiated away nor work is done upon the electron by the external forces. How to understand this paradoxical situation?
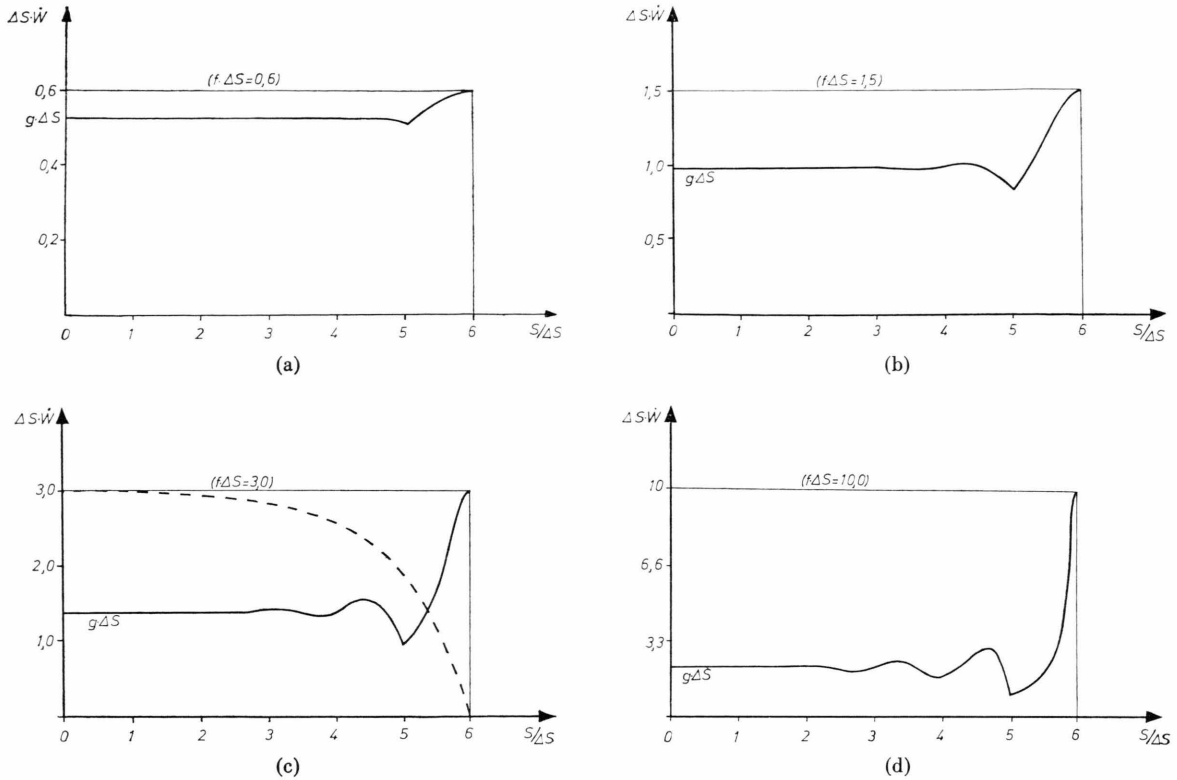
Fig. 1. The invariant acceleration $\dot{w}$ (dimensionless: $\Delta s \cdot \dot{w}$) is plotted versus proper time s (dimensionless: $s/\Delta s$) for various constant forces f (dimensionless: $f \cdot \Delta s$). With increasing force f there arise more and more conspicuous oscillations in characteristic time intervals prior to the end of the force. Dotted curve: Lorentz-Dirac acceleration (without pre-acceleration).

It seems to us that a possible resolution of this problem can be achieved, if one refers to the notion of an extended particle. The world-line, which has to be calculated from the equation of motion (II, 7), is then to be considered as the world-line of the (eventually fictive) center of the particle. The surroundings of this center, which must be defined in a retarded way (see Ref. [9]), carry the field energy corresponding to the mass of the particle (we think of a purely electromagnetic electron). Since the particle as a whole interacts with an external force field by means of the center alone (usual Lorentz force only slightly modified), the changes of the state of motion of the center cannot be transmitted instantaneously to the surroundings, where the inertia of the particle is located. For instance, in our earlier cut-off theory [9] formula (24) for the calculation of the bound four-momentum of the particle

$$P_{\mathrm{b}}{}^{\mu} = \frac{1}{c} \frac{Z^2}{8\pi} \int \mathrm{d}\hat{\Omega}\, \hat{n}^{\mu} \int_{R_{\min}}^{\infty} \frac{\mathrm{d}R}{R^2} = \frac{1}{c} \frac{Z^2}{8\pi} \int_{\hat{\Omega}} \mathrm{d}\hat{\Omega}\, \frac{\hat{n}^{\mu}}{R_{\min}} = \frac{1}{c} \frac{Z^2}{2\varrho} \left\{ \frac{3}{4}\, \hat{u}^{\mu}(u\,\hat{u}) - \frac{1}{3}\, u^{\mu} \right\} \qquad \text{(III, 6)}$$

shows explicitly that the field energy (and therefore also the inertia) is concentrated mainly on a spherical shell in the rest system of the center and that the dimensions of this shell are in the order of the classical electron radius.

According to these ideas, the particle gains no energy nor momentum in the first Δs-interval, which

can be easily seen from formula (V, 2b) of the earlier work [4], where one has found for the bound energy-momentum *

$$P_{\mathrm{b}}{}^{\mu}{}_{(s)} = m\,c\,u^{\mu}{}_{(s-\Delta s)}\,. \qquad \text{(III, 8)}$$

* Because of the discrepancy between expressions (III, 6) and (III, 8) consult reference [12].

Therefore, in the first interval the energy and momentum are not altered with respect to that state of motion, which the particle has occupied before the force has been switched-on, though the center is clearly accelerated in the time interval under consideration.

The same arguments apply to the radiated energy (III, 4'): Because of the relativistic velocity-addition theorem, which can be written for the linear motion considered here as

$$v_{\text{rel}}/c = \frac{v_{(s)}/c - v_{(s-\Delta s)}/c}{1 - \dfrac{v_{(s)}}{c} \cdot \dfrac{v_{(s-\Delta s)}}{c}} = \text{Tanh} \left[w_{(s)} - w_{(s-\Delta s)} \right] \tag{III, 9}$$

the radiated energy per unit time is

$$dE_{\text{rad}}/dt = c \, K_{(s-\Delta s)} \cdot \frac{v_{\text{rel}}}{c} . \tag{III, 4''}$$

Only if the particle's center has gained a relative velocity v_{rel} different from zero during the proper time $\Delta s/c$ has elapsed, radiation is produced and running out as a disturbance of the Coulomb field (cf. the book of Sexl and Urbantke[10], p. 97). But since this disturbance, produced properly on the world line of the center, takes the time $\Delta s/c$ to escape through the surface of the particle, the outgoing radiation cannot be found in the outside region of the particle until the above mentioned time interval has elapsed. Hence, there is no radiation in the first Δs-interval after the force has been switched-on.

Quite analogous arguments are assumed to hold for the switching-off of the force at s_{fin} ($= 6\,\Delta s$) (see Fig. 1), where, e.g., radiation is still emitted after the center moves already with constant four-velocity.

β) Causality-violating Self-oscillations

It is true, the discontinuity of the acceleration $\dot{w}$ at $s = 0$, where the force also exhibits a discontinuity, is exactly that which one expects in the case of an everlasting constant force of equal magnitude. The corresponding acceleration $\dot{w} = g$ ($=$ const) is to be determined from the equation of motion

$$m \, c^2 \, g \, \text{Cosh} \, (g \, \Delta s) = K \, (= \text{const}) . \tag{III, 10}$$

But when time goes on, the particle excites itself to oscillate around this fixed value $g_{(K)}$ of the acceleration until the latter reaches the value of the neutral particle

$$m \, c^2 \, g^{np} = K \tag{III, 11}$$

in $s = s_{\text{fin}}$, where the force drops down to zero. These self-oscillations, being completely determined by the external force, arise in a characteristic time interval, which depends upon the magnitude of the force, before the force ends to act on the particle, and the particle is thus able to "feel" in advance the end of the force approaching. Since the particle under consideration constitutes an open system, this phenomenon is clearly a violation of causality[11].

In the following, we want to analyze these causality violating effects in greater detail, and the adequate means to do this seems to be a linearization of the non-linear equation of motion, where one shall be able to describe the oscillations by harmonic functions with exponentially increasing amplitude, such as $e^{as} \sin(b \, s)$. This procedure is suggested by the regular shape of the oscillations (see Figure 2). Observe, that the discontinuities of the higher derivatives of $\dot{w}_{(s)}$, which are due to the discontinuity of the force, are smoothed with increasing distance to the end of the force. Especially, we are interested in the damping constant a as function of the force strength K.
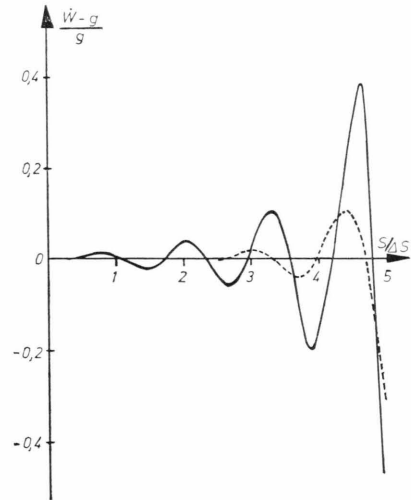

Fig. 2. The oscillations in relative units $g^{-1}(w-g)$ for $(f \cdot \Delta s) = 10$ (solid curve) and $(f \cdot \Delta s) = 3$ (dashed curve). For stronger forces the oscillations arise earlier: the effect of causality violation becomes worse.

IV. Linearized Treatment of Oscillations

In order to come to a linearized treatment of the causality-violating oscillations let us start with the

integral Eq. (II, 8) together with the special force (III, 1)

$$\Phi_{(s)} = f \int\limits_{s'=s}^{s+\Delta s} \frac{ds'}{\mathrm{Cosh}\,\Phi_{(s')}} \, . \qquad (IV, 1)$$

Now we try the ansatz

$$\Phi_{(s)} = \Phi_0 + \varepsilon_{(s)}\,; \quad |\varepsilon_{(s)}| \ll |\Phi_0| \qquad (IV, 2)$$

and linearize in the following way

$$\mathrm{Cosh}\,\Phi_{(s)} = \mathrm{Cosh}\,[\Phi_0 + \varepsilon_{(s)}]$$
$$\approx \mathrm{Cosh}\,\Phi_0 \{1 + \varepsilon_{(s)}\,\mathrm{Tanh}\,\Phi_0\}\, . \qquad (IV, 3)$$

Hence, the linearized equation of motion (IV, 1) becomes

$$\Phi_0 + \varepsilon_{(s)} = f \int\limits_{s}^{s+\Delta s} \frac{ds'}{\mathrm{Cosh}\,\Phi_0} - f\,\frac{\mathrm{Tanh}\,\Phi_0}{\mathrm{Cosh}\,\Phi_0} \int\limits_{s}^{s+\Delta s} \varepsilon_{(s')}\,ds'\, . \qquad (IV, 4)$$

Separating here the constant from the varying terms yields

$$\Phi_0 = f \int\limits_{s}^{s+\Delta s} \frac{ds'}{\mathrm{Cosh}\,\Phi_0} = \frac{f\,\Delta s}{\mathrm{Cosh}\,\Phi_0} \qquad (IV, 5)$$

and

$$\varepsilon_{(s)} = -\,\Phi_0\,\mathrm{Tanh}\,\Phi_0 \int\limits_{s}^{s+\Delta s} \varepsilon_{(s')}\,\frac{ds'}{\Delta s}\, , \qquad (IV, 6)$$

where we have used the fact that $\Phi_0\,(= g \cdot \Delta s)$ satisfies the equation of motion (III, 10).

In the following we shall investigate the linear integral Eq. (IV, 6), but before doing this we have to show the connection between $\varepsilon_{(s)}$ and the acceleration $\dot{w}_{(s)}$. Decomposing $\dot{w}_{(s)}$ in its constant and oscillatory part $(|\delta\dot{w}| \ll g)$

$$\dot{w}_{(s)} = g + \delta\dot{w}_{(s)} \qquad (IV, 7)$$

and using the linearization procedure (IV, 3) yields at once

$$\Delta s\,\delta\dot{w}_{(s)} = -\,\varepsilon_{(s)}\,\Phi_0\,\mathrm{Tanh}\,\Phi_0\, , \qquad (IV, 8)$$

so it is sufficient to study the oscillations of $\varepsilon_{(s)}$. Abbreviating

$$\lambda^2 := \Phi_0\,\mathrm{Tanh}\,\Phi_0\,; \quad \lambda^2 \geqq 0 \qquad (IV, 9)$$

and trying in (IV, 6) the ansatz

$$\varepsilon_{(s)} \sim \exp\{\sigma\,s/\Delta s\} \qquad (IV, 10)$$

leads to the transcendental equation for σ

$$\sigma = -\,\lambda^2 (e^\sigma - 1)\, . \qquad (IV, 11)$$

From this equation it is easily seen that σ cannot be a real number. For, if this would be the case, the

right-hand side of Eq. (IV, 11) would always have the opposite sign of the left-hand side. Of course, $\sigma = 0$ is a trivial solution. Since σ must be a complex number, we put

$$\sigma = a + i\,b\,; \quad a, b | = \mathrm{real}\,, \qquad (IV, 12)$$

and split up Eq. (IV, 11) into two equations

$$a = -\,\lambda^2 (e^a \cos b - 1)\, , \qquad (IV, 13a)$$
$$b = -\,\lambda^2 e^a \sin b\, , \qquad (IV, 13b)$$

which means that the ansatz (IV, 10) is equivalent to the fundamental system of oscillatory exponential solutions

$$\varepsilon_{(s)} \sim \exp\{a\,s/\Delta s\} \cdot \begin{cases} \sin(b\,s/\Delta s) \\ \cos(b\,s/\Delta s)\, . \end{cases} \qquad (IV, 14)$$

Now we proceed to show that the system of Eqs. (IV, 13) has an enumerably infinite set of solutions (a_n, b_n) with $n = 1, 2, 3, \ldots$, and that $a_n > a_m$ if $n > m$. So we shall confine ourselves to the discussion of a_1 when studying the causality violation effects of greatest extent.

First, one concludes from (IV, 13a) that the "damping" constant a must always be a positive number: $a_n > 0$. Next, one finds from (IV, 13b) that a sign-reversed b_n is also a solution. Hence, we can restrict ourselves to positive b_n. Furthermore,
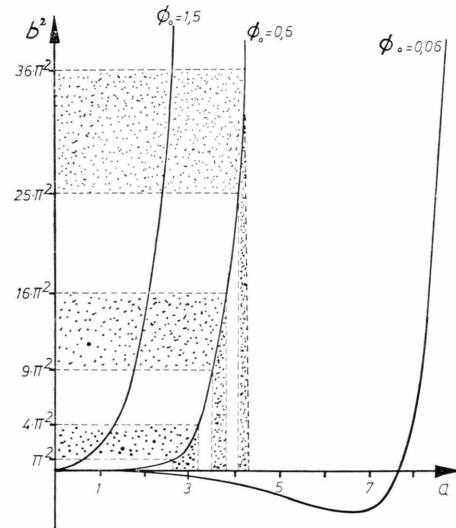

Fig. 3. Eliminating the angular functions from the system of Eq. (IV, 13) yields $b^2 = \lambda^4 e^{2a} - (\lambda^2 - a)^2$. Since possible solutions for b must be situated in the intervals (IV, 15), the damping constants a_n increase in magnitude with increasing order of the intervals (IV, 15).

the possible values of b_n are restricted to the intervals

$$(2n-1)\pi < b_n < 2n\pi; \quad n = 1, 2, 3, \ldots \quad (IV, 15)$$

in order that $\sin b_n < 0$ (for positive b_n).

These two conclusions are sufficient to recognize that possible solutions of (IV, 13) for the damping constant a form a discrete set $\{a_n; n = 1, 2, 3, \ldots\}$ which contains a minimal number a_1 (see Figure 3). In order to get an impression of the dependence of the pair (a_n, b_n) upon the external force f, which enters the Eq. (IV, 13) through (IV, 9) with (IV, 5), one can transform the two Eqs. (IV. 13) into the following set

$$a_I(b) = -\ln \lambda^2 - \ln |\sin b/b| \quad (IV, 16a)$$

$$a_{II}(b) = \lambda^2 - b \cos b / |\sin b|. \quad (IV, 16b)$$

The solutions $a_n(\lambda)$ and $b_n(\lambda)$ of (IV, 13) are given by the intersections of the two curves (IV, 16) in a
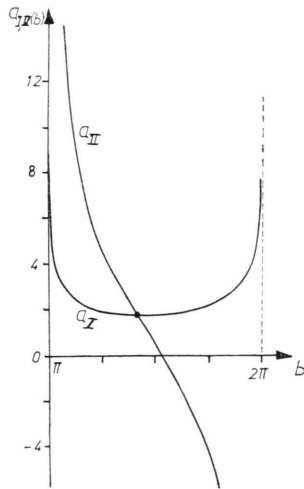
Fig. 4. Intersection of the curves (IV, 16) in the first of the intervals (IV, 15). There is only one intersection in each interval.

** We are not interested here in the determination of the amplitudes of the various modes (a_n, b_n) of the fundamental system (IV, 14). This is clearly a problem in itself, but qualitatively one can say (see Fig. 2) that the relative amplitude $(\dot{w} - g/g)$ increases with increasing force K, and therefore the causality violation effect is enhanced by the force dependence of the amplitudes!

(a/b)-diagram (Figure 4). The qualitative shape of the two curves is the same in all intervals (IV, 15). If the force parameter λ varies, the two curves are moved vertically as a whole without changing their profile. With decreasing λ, curve (b) moves downwards and curve (a) moves upwards. Hence, the point of intersection moves to the left

$$[b_n \to (2n-1)\pi]$$

but simultaneously the value of the damping constant increases unboundedly $(a_n \to \infty)$. Conversely, if the force grows without limitation $(f \to \infty \Rightarrow \lambda \to \infty)$, the point of intersection moves to the right $(b_n \to 2n\pi; \ a_n \to 0)$. Figure 5 shows the numerical
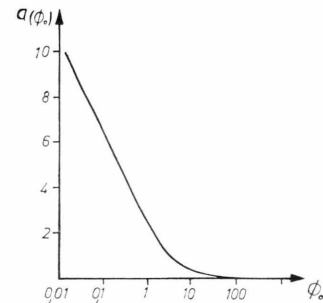
Fig. 5. The smallest damping constant a_1 as function of the force parameter Φ_0 [see Eq. (IV, 5)]. The relation between the force f and parameter Φ_0 $(= g \cdot \Delta s)$ is given in more detail in reference [4].

solution for $a_1(\lambda)$. Since the general solution of the linear Eq. (IV, 6) is a superposition of all possible modes (a_n, b_n) of the fundamental system (IV, 14), the effect of causality violation is chiefly ** determined by the first mode (a^1, b_1), which exhibits the smallest value of the damping constant a. But a_1 tends to zero for ever-increasing force $(f \to \infty \Rightarrow \Phi_0 \to \infty)$, and therefore the causality violation effect reaches back to the arbitrarily distant past. However, for reasonable forces $(\Phi_0 \approx f \Delta s \ll 1)$ the magnitude of causality violation becomes arbitrarily small $(a_1 \to \infty)$. This is in line with the general results of Section II.

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